

Unit-1 Ordinary Differential Equation.

Differential eqⁿ - An eqⁿ which contains the derivatives of dependent variable w.r. to independent variable, dependent variable & independent variables is called differential eqⁿ. It can be classified into 2 types.

1. Ordinary Differential Equation. (O.D.E)
2. Partial Differential Equation (P.D.E)

Ordinary Differential Equation.

A differential equation in which derivatives of dependent variable w.r. to only one independent variable are present, that type of differential eqⁿ is called O.D.E.

$$\text{eg} \rightarrow \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$$

Partial Differential Equation.

An eqⁿ which contains the partial derivatives of dependent variable w.r. to 2 or more than 2 independent variables are called partial differential eqⁿ.

$$\text{eg} \rightarrow \frac{\partial^2 z}{\partial x^2} + 5 \frac{\partial^2 z}{\partial x \partial y} + 3 \frac{\partial z}{\partial x} = z$$

Order of a differential Equation.

The order of a diff. eqⁿ is the order of the highest derivative present in the equation.

Eg $\rightarrow \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$ Order = 2.

(2) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 = y$ Order = 3

Degree of a Differential Equation.

The degree of a diff. eq.ⁿ is the power of highest derivative present in the eq.ⁿ when, when it is made free from radical sign & fractional powers in terms of derivatives.

Eg $\rightarrow \sqrt{1 + \frac{dy}{dx}} = \frac{d^2y}{dx^2}$ Order = 2
Degree = 2.

(2) $\left(\frac{d^2y}{dx^2}\right)' + 3\left(\frac{dy}{dx}\right)^5 + y^{1/3} = x^{2/3}$ Order = 2
Degree = 1.

Linear Differential Equation.

A D.E in dependent variable 'y' is said to be linear, if y & all its derivatives occurs in its first degree & are not multiplied together.

Eg $\rightarrow$ (1) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^5 + y^{1/3} = x^{2/3}$ is non linear eq.ⁿ.

(2) $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right) + 3y = x^5$ is linear equation

(3) $\frac{d^3y}{dx^3} + 5\left(y \cdot \frac{d^2y}{dx^2}\right) + \frac{dy}{dx} = y$ is a non-linear equation

Leibnitz's Linear Differential Equation.

An eqⁿ of the form, $\frac{dy}{dx} + Py = Q$ is called Leibnitz's L.D.E.

where P & Q are function of x , to solve this eqⁿ we first find I.F = $e^{\int P dx}$. Soluⁿ is given by.

$$y \times I.F = \int Q \times (I.F) dx + c.$$

Q1 Solve the D.E, $\frac{dy}{dx} + 2y = x$, $P=2$, $Q=x$

$$I.F = e^{\int P dx} = e^{\int 2 dx} = e^{2x}$$

$$y \times e^{2x} = \int x \times e^{2x} dx + c.$$

$$y \times e^{2x} = \frac{x \cdot e^{2x}}{2} - \int \frac{1 \cdot e^{2x}}{2} dx + c$$

$$y \times e^{2x} = \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + c.$$

Ans.

Q2

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

$$\frac{dy}{dx} + \sec^2 x \cdot y = \frac{\sin x}{\cos^3 x}$$

$$P = \sec^2 x \quad Q = \frac{\sin x}{\cos^3 x}$$

$$I.F = e^{\int P dx} = e^{\int \sec^2 x} = e^{\tan x}$$

$$y \times e^{\tan x} = \int \frac{\sin x}{\cos^3 x} \times e^{\tan x} dx + c.$$

Linear differential equation of n^{th} order with constant coefficients

An equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = \phi(x)$$

where a_0, a_1, a_2 are all constant and $\phi(x)$ is a function of x is called linear differential equation of n^{th} order with constant coefficients.

$$\text{Let } \frac{d}{dx} = D$$

$$\Rightarrow a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = \phi(x)$$

$$\Rightarrow (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = \phi(x). \text{--- (1)}$$

$$\Rightarrow \boxed{F(D) y = \phi(x)}$$

It can be classified into two types

1. Homogenous $\rightarrow$ This differential equation will be called homogenous if $\phi(x) = 0$

2. Non Homogenous - This differential equation will be called non homogenous if $\phi(x) \neq 0$

The complete solution of differential eqⁿ is given by
 $y = \text{Complementary function} + \text{Particular Integral}$

$$\boxed{y = C.F + P.I}$$

where, CF is the general solution of corresponding homogenous differential eqⁿ $F(D)y = 0$ &

PI is a particular solution of given differential equation

Rules for finding C.F

To find C.F, we first find Auxilliary Equation (A.E) by replacing D by m & y by 1 in eqⁿ ① we get,

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0$$

Now, C.F depends upon the roots of the auxilliary equation.

① — Case I → When roots of Auxilliary eqⁿ are real & different

Let, $m_1, m_2, m_3, \dots, m_n$ are roots

$$C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \dots + C_n e^{m_n x}$$

Case-2 - When two real roots are equal

Let, $m_1 = m_2$

$$C.F = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + \dots + C_n e^{m_n x}$$

Case 3 - When three roots are equal

Let, $m_1 = m_2 = m_3$

$$C.F = (C_1 + C_2 x + C_3 x^2) e^{m_1 x} + C_4 e^{m_4 x} + \dots + C_n e^{m_n x}$$

Case 4 - When two roots of A.E. are complex no.
Let, $\alpha \pm i\beta$ are roots.

$$C.F = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$$

Case 5 - When two complex roots are equal.

Let, $\alpha \pm i\beta, \alpha \pm i\beta$ are roots.

$$C.F = e^{\alpha x} [(C_1 + C_2 x) \cos \beta x + (C_3 + C_4 x) \sin \beta x]$$

Case 6 - When roots of A.E. are irrational no.
Let, $\alpha \pm \sqrt{\beta}$ are roots.

$$C.F = e^{\alpha x} [C_1 \cosh \sqrt{\beta} x + C_2 \sinh \sqrt{\beta} x]$$

Case 7 -> When two irrational roots are equal.

Let $\alpha \pm \sqrt{\beta}, \alpha \pm \sqrt{\beta}$ are roots.

$$C.F = e^{\alpha x} [(C_1 + C_2 x) \cosh \sqrt{\beta} x + (C_3 + C_4 x) \sinh \sqrt{\beta} x]$$

Q1 Solve; $\frac{d^2 y}{dx^2} - 7 \frac{dy}{dx} + 12y = 0$

Let, $\frac{d}{dx} = D$

$$D^2 y - 7Dy + 12y = 0$$

$$(D^2 - 7D + 12)y = 0$$

gt's A.E.

$$D \rightarrow m, y \rightarrow 1$$

$$m^2 - 7m + 12 = 0$$

$$(m-3)(m-4) = 0$$

$$m = 3, 4$$

$$C.F = C_1 e^{3x} + C_2 e^{4x}$$

$$P.I = 0$$

Complete solun.

$$y = C.F + P.I$$

$$y = C_1 e^{3x} + C_2 e^{4x}$$

Ans

Q2

Solve,

$$\frac{d^4 y}{dx^4} - n^4 y = 0$$

Let, $\frac{d}{dx} = D$

$$D^4 y - n^4 y = 0$$

$$(D^4 - n^4)y = 0$$

$$D \rightarrow m, y \rightarrow 1$$

$$(m^4 - n^4) = 0$$

$$\Rightarrow (m^2 - n^2)(m^2 + n^2) = 0$$

$$\Rightarrow (m-n)(m+n)(m^2 + n^2) = 0$$

$$\Rightarrow m = n, -n, \pm ni$$

$$\Rightarrow m = n, -n, 0 \pm ni$$

$$m^2 + n^2 = 0$$

$$m^2 = -n^2$$

$$\therefore m = \pm \sqrt{-n^2}$$

$$m = \pm ni$$

$$C.F = C_1 e^{nx} + C_2 e^{-nx} + e^{0x} [C_3 \cos nx + C_4 \sin nx]$$

$$P.I = 0.$$

$$\text{Comp. solu}^n \Rightarrow y = C.F + P.I$$

$$y = C_1 e^{nx} + C_2 e^{-nx} + e^{0x} [C_3 \cos nx + C_4 \sin nx]$$

Q.

$$\text{Solve } \frac{d^5 y}{dx^5} - \frac{d^3 y}{dx^3} = 0$$

$$\text{Let } \frac{d}{dx} = D$$

$$(D^5 - D^3)y = 0$$

$$D \rightarrow m, y \rightarrow 1$$

$$(m^5 - m^3) = 0$$

$$m^3(m^2 - 1) = 0$$

$$m = 0, 0, 0, +1, -1$$

$$C.F = (C_1 + C_2 x + C_3 x^2) e^{0x} + C_4 e^x + C_5 e^{-x}$$

$$P.I = 0$$

$$y = C.F + P.I$$

$$y = C_1 + C_2 x + C_3 x^2 + C_4 e^x + C_5 e^{-x}$$

Q. Solve, $\frac{d^6 y}{dx^6} - y = 0$

Let, $\frac{d}{dx} = D$.

$$D^6 y - y = 0.$$

$D \rightarrow m, y \rightarrow 1.$

$$y(D^6 - 1) = 0$$

$$m^6 - 1 = 0$$

$$(m^3 - 1)(m^3 + 1) = 0$$

$$(m-1)(m^2+1+m)(m^3+1) = 0.$$

$$(m-1)(m^2+m+1)(m^3+1)(m^2+m+1)$$

Continue

H.W

$$\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = 0.$$

Let, $\frac{d}{dx} = D$

$$D^2 y - 3Dy + 2y = 0$$

$$(D^2 - 3D + 2)y = 0$$

$D \rightarrow m, y \rightarrow 1.$

$$m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$m(m-2) - 1(m-2) = 0$$

$$m = 1, 2.$$

$$C.F = 1x e^x + 2e^{2x}$$

P.I = 0 $\rightarrow$ Homogeneous eqⁿ.

Soluⁿ,

$$y = C.F + P.I = e^x + 2e^{2x}.$$

Ans

Q3

$C_1 + (C_2 + C_3)e^{-x}$ is C.F.

of $\frac{d^3 y}{dx^3} + A \frac{d^2 y}{dx^2} + \frac{dy}{dx} = x$

Then, $A = ?$

$$C_1 e^{0 \cdot x} + (C_2 + C_3) e^{-x}$$

Continue.

$$m = 1, -1, m = \frac{-1 \pm \sqrt{1-4}}{2}, m = \frac{1 \pm \sqrt{1-4}}{2}$$

$$m = 1, -1, \frac{-1 \pm \sqrt{3}i}{2}, \frac{1 \pm \sqrt{3}i}{2}$$

$$\begin{aligned} \text{C.F.} = & C_1 e^x + C_2 e^{-x} + e^{-\frac{1}{2}x} \left[C_3 \cos \frac{\sqrt{3}x}{2} + C_4 \sin \frac{\sqrt{3}x}{2} \right] \\ & + e^{\frac{1}{2}x} \left[C_5 \cos \frac{\sqrt{3}x}{2} + C_6 \sin \frac{\sqrt{3}x}{2} \right] \end{aligned}$$

II

$$(D^6 + 1)y = 0$$

$$m^6 + 1 = 0$$

$$(m^2)^3 + 1 = 0$$

$$(m^2 + 1)(m^4 - m^2 + 1) = 0$$

Consider the differential eqⁿ.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = \phi(x)$$

$$F(D)y = \phi(x)$$

The P.I of above differential eqⁿ is given by

$$P.I = \frac{1}{F(D)} \phi(x)$$

Case I - when $\phi(x) = e^{ax+b}$

P.I.

$$\frac{1}{F(D)} e^{ax+b} = \frac{1}{F(a)} e^{ax+b} \quad \text{if } F(a) \neq 0$$

if $F(a) = 0$.

$$\text{Then, } \frac{1}{F(D)} e^{ax+b} = x \times \frac{1}{F'(a)} e^{ax+b} \quad \text{if } F'(a) \neq 0$$

if $F'(a) = 0$

$$\text{Then, } \frac{1}{F(D)} e^{ax+b} = x^2 \times \frac{1}{F''(a)} e^{ax+b} \quad \text{if } F''(a) \neq 0$$

So on.

Q. Solve the differential eqⁿ, $\frac{d^3 y}{dx^3} + y = 3 + 5e^x$.

$$\text{let, } \frac{d}{dx} = D.$$

$$D^3 y + y = 3 + 5e^x.$$

$$y(D^3 + 1) = 3 + 5e^x.$$

$$\text{A.E, } D \rightarrow m, y \rightarrow 1.$$

$$m^3 + 1 = 0.$$

$$(m+1)(m^2 - m + 1) = 0$$

$$m = -1, m = \frac{1 + \sqrt{1-4}}{2}.$$

$$m = -1, m = \frac{1 \pm i\sqrt{3}}{2}.$$

$$m = -1, \frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}.$$

$$C.F = C_1 e^{-x} + e^{\frac{1}{2}x} \left[C_2 \cos \frac{\sqrt{3}}{2} x + C_3 \sin \frac{\sqrt{3}}{2} x \right].$$

$$P.I = \frac{1}{D^3 + 1} (3 + 5e^x).$$

$$= 3 \times \frac{1}{D^3 + 1} x e^{0 \cdot x + 0} + 5 \times \frac{1}{D^3 + 1} x e^x.$$

$$= 3 \times \frac{1}{0+1} x + 5 \times \frac{1}{1+1} x e^x.$$

$$= 3 + \frac{5}{2} e^x.$$

Complete solnⁿ, $y = C.F + P.I.$

$$y = C_1 e^{-x} + e^{\frac{1}{2}x} \left[C_2 \cos \frac{\sqrt{3}}{2} x + C_3 \sin \frac{\sqrt{3}}{2} x \right] + 3 + \frac{5}{2} e^x$$

Ans

Q. Solve the differential eqⁿ.

$$\frac{d^3 y}{dx^3} - 3 \frac{dy}{dx} + 2y = e^{-2x} + 2 \sinh x.$$

$$D^3 y - 3Dy + 2y = 0$$

$$m^3 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2.$$

(m=1) satisfy the eqⁿ

so (m-1) is its factor

$$(m-1)(m^2 + m - 2) = 0$$

$$(m-1)(m-1)(m+2) = 0$$

$$m = 1, 1, -2$$

$$C.F = (c_1 + c_2 x)e^x + c_3 e^{-2x}$$

$$P.I = \frac{1}{D^3 - 3D + 2} (e^{-2x} + 2 \sinh x)$$

$$D^3 - 3D + 2$$

$$= \frac{1}{D^3 - 3D + 2} [e^{-2x} + 2(e^x - e^{-x})]$$

$$= \frac{1}{D^3 - 3D + 2} x e^{-2x} + \frac{1}{D^3 - 3D + 2} e^x - \frac{1}{D^3 - 3D + 2} x e^{-x}$$

$$= \frac{2x}{3D^2 - 3} e^{-2x} + x \times \frac{1}{3D^2 - 3} x e^x - \frac{1}{-1 + 3 + 2} e^{-x}$$

$$\frac{1}{4} + 2 \times \frac{1}{2} + 1$$

$$= x \times \frac{1}{3 \times 4 - 3} e^{-2x} + x^2 \times \frac{1}{6D} \times e^x - \frac{1}{4} e^{-x}$$

$$P.I = x \times \frac{1}{9} e^{-2x} + x^2 \times \frac{1}{6} e^x - \frac{1}{4} e^{-x}$$

$$y = C.F + P.I.$$

Q. $4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + y = 4e^{-x/2}$

$$(4D^2 + 4D + 1)y = 0$$

$$4m^2 + 4m + 1 = 0$$

$$D = -16 - 2m(2m+1) \quad 4m^2 + 2m + 2m + 1 = 0$$

$$2m(2m+1) + 1(2m+1) = 0$$

$$m = -\frac{1}{2}, -\frac{1}{2}$$

$$C.F = (C_1 + C_2 x) e^{-\frac{1}{2}x}$$

$$P.I = \frac{1}{F(D)} e^{ax+b} = \frac{1}{F(a)} e^{ax+b}$$

$$= \frac{4 \times 1}{4D^2 + 4D + 1} \times e^{-\frac{1}{2}x} =$$

$$P.I = x \times \frac{4 \times e^{-\frac{1}{2}x}}{8D + 4}$$

$$= x \times \frac{4 \times e^{-\frac{1}{2}x}}{8D + 4} = \frac{x^2 \times 4 \times e^{-\frac{1}{2}x}}{8 \times 2}$$

$$P.I = \frac{x^2 e^{-\frac{1}{2}x}}{2}$$

Case II $\rightarrow$ when $\phi(x) = \sin(ax+b)$ OR $\cos(ax+b)$

P.I.

$$\frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-a^2)} \times \sin(ax+b) \quad \text{if } F(-a^2) \neq 0.$$

$$\text{if } F(-a^2) = 0 \quad D^2 \rightarrow -a^2$$

$$\text{then, } \frac{1}{F(D^2)} \sin(ax+b) = x \times \frac{1}{F'(-a^2)} \sin(ax+b) \quad \text{if } F'(-a^2) \neq 0$$

or so on.

Same for $\cos(ax+b)$.

Q. Solve $\frac{d^2 y}{dx^2} + 4y = \sin 3x + \cos 2x$.

C.F. $(D^2 + 4)y = 0$.

$D^2 = -4$

$m^2 + 4 = 0$.

$m^2 = -4$

$m = \pm 2i$

C.F. $= e^{0x} [C_1 \cos \beta x + C_2 \sin \beta x]$

C.F. $= e^{0x} [C_1 \cos 2x + C_2 \sin 2x] \quad \text{--- (1)}$

P.I. $= \frac{1}{D^2 + 4} (\sin 3x + \cos 2x)$

$= \frac{1}{D^2 + 4} \sin 3x + \frac{1}{D^2 + 4} \cos 2x$

$$= \frac{1}{-9+4} \sin 3x + 2x \cdot \frac{1}{2D} \times \cos 2x$$

$$= \frac{1}{-5} \sin 3x + x \times \frac{1}{2} \int \cos 2x dx$$

$$= -\frac{1}{5} \sin 3x + \frac{x}{2} \times \frac{\sin 2x}{2}$$

$$P.I = -\frac{1}{5} \sin 3x + \frac{x}{4} \sin 2x$$

Complete solnⁿ.

$$y = C.F + P.I.$$

$$y =$$

Q.

Solve, $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + y = \cos x \cdot \cos 2x$

C.F, $D^2 y - 4Dy + y = 0$

$$y(D^2 - 4D + 1) = 0$$

$$D \rightarrow m$$

$$y \rightarrow 1$$

$$m^2 - 4m + 1 = 0$$

$$m = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{2 \pm \sqrt{3}}{1}$$

$$m = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$C.F = e^{2x} [C_1 \cosh \sqrt{3}x + C_2 \sinh \sqrt{3}x]$$

$$P.I = \frac{1}{D^2 - 4D + 1} \times \cos x \cdot \cos 2x \times \frac{1}{2}$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - 4D + 1} (\cos 3x + \cos(-x)) \right]$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - 4D + 1} \cos 3x + \frac{1}{D^2 - 4D + 1} \cos x \right]$$

$$= \frac{1}{2} \left[\frac{1}{-9 - 4D + 1} \cos 3x + \frac{1}{-1 - 4D + 1} \cos x \right]$$

$$= \frac{1}{2} \left[\frac{1}{-4D - 8} \cos 3x + \frac{1}{-4D} \cos x \right]$$

$$= \frac{1}{2} \left[\frac{1}{-4} \times \frac{1}{D+2} \cos 3x - \frac{1}{4} \times \frac{1}{D} \cos x \right]$$

$$= \frac{1}{2} \left[-\frac{1}{4} \times \frac{D-2}{(D+2)(D-2)} \cos 3x - \frac{1}{4} \int \cos x \, dx \right]$$

$$= \frac{1}{2} \left[-\frac{1}{4} \frac{(D-2)}{D^2 - 4} \cos 3x - \frac{1}{4} \sin x \right]$$

$$= \frac{1}{2} \left[-\frac{1}{4} \times \frac{D-2}{-9-4} \cos 3x - \frac{1}{4} \sin x \right]$$

$$= \frac{1}{2} \left[\frac{1}{52} (D \cos 3x - 2 \cos 3x) - \frac{1}{4} \sin x \right]$$

$$P.I = \frac{1}{2} \left[\frac{1}{52} (-3 \sin 3x - 2 \cos 3x) - \frac{1}{4} \sin x \right]$$

comp. soln, $y = C.F + P.I$

$y =$

Ans

$$\cos 2x = 2 \cos^2 \frac{x}{2} - 1$$

Q. Solve the differential eqⁿ

$$\frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 2y = e^x + 2 \cos^2 \frac{x}{2}$$

$$\text{Let } \frac{d}{dx} \rightarrow D$$

$$D^3 y - 3D^2 y + 4Dy - 2y = e^x + 2 \cos^2 \frac{x}{2}$$

$$D \rightarrow m, y \rightarrow 1$$

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$\begin{array}{r} (m-1) \cdot m^3 - 3m^2 + 4m - 2 \quad | \quad m^2 - 2m + 2 \\ \underline{-m^3 + m^2} \\ -2m^2 + 4m \\ \underline{+ 2m^2 - 4m} \\ 2m - 2 \\ \underline{- 2m + 2} \\ 0 \end{array}$$

$$(m-1)(m^2 - 2m + 2) = 0$$

$$m = 1, \quad \frac{2 \pm \sqrt{4-8}}{2}$$

$$m = 1, \quad 1 \pm i$$

$$C.F = c_1 e^x + e^x [c_2 \cos x + c_3 \sin x] \quad \text{--- (1)}$$

$$P.I = \frac{1}{D^3 - 3D^2 + 4D - 2} \times \left[e^x + 2 \cos^2 \frac{x}{2} \right]$$

$$= \frac{1}{D^3 - 3D^2 + 4D - 2} \times e^x + \frac{1}{D^3 - 3D^2 + 4D - 2} \times (\cos x + 1)$$

$$= \frac{x \times 1}{3D^2 - 6D + 4} \times e^x + \frac{1}{D^3 - 3D^2 + 4D - 2} \times \cos x + \frac{1}{D^3 - 3D^2 + 4D - 2} \times e^{0 \cdot x}$$

$$= x e^x + \frac{1}{-2} e^{0 \cdot x} + \frac{1}{D^2 x D - 3D^2 + 4D - 2} \cos x$$

$$= x e^x - \frac{1}{2} + \frac{1}{-D + 3 + 4D - 2} x \cos x$$

$$= x e^x - \frac{1}{2} + \frac{1}{3D + 1} x \cos x$$

$$= x e^x - \frac{1}{2} + \cos x \times \frac{(3D - 1)}{(3D + 1)(3D - 1)}$$

$$= x e^x - \frac{1}{2} + \frac{3D \cos x - \cos x}{9D^2 - 1}$$

$$= x e^x - \frac{1}{2} + \frac{3(-\sin x) - \cos x}{10}$$

$$P.I = x e^x - \frac{1}{2} + \frac{3 \sin x + \cos x}{10}$$

Comp. soluⁿ

$$y = C.F + P.I$$

Case III $\rightarrow$ When $\phi(x) = x^m$

where, m is positive integer.

$$P.I = \frac{1}{F(D)} x^m$$

Step-1 Take common the lowest degree term from $F(D)$ so that remaining factor takes the form $[1 + \phi(D)]$ or $[1 - \phi(D)]$

Step-2. Take this factor into numerator, it will take the form $[1 + \phi(D)]^{-1}$ or $[1 - \phi(D)]^{-1}$.

Step-3. Expand it by binomial series & operate x^m term by term.

Note:

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

Q.1 solve

$$\frac{d^3 y}{dx^3} + 8y = x^4 + 2x + 1$$

$$\text{let } \frac{d}{dx} = D$$

$$D^3 y + 8y = 0$$

$$D \rightarrow m, \quad y \rightarrow 1.$$

$$m^3 + 8 = 0$$

$$m^3 + 2^3 = 0$$

$$(m+2)(m^2 + 4 + 2m) = 0.$$

$$(m+2)(m^2 - 2m + 4) = 0$$

$$m = -2, \quad \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$m = -2, \quad \frac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}i$$

$$m = -2, 1 \pm \sqrt{3}i$$

$$C.F = C_1 e^{-2x} + e^x [C_2 \cos \sqrt{3}x + C_3 \sin \sqrt{3}x]$$

$$P.I = \frac{1}{F(D)} \phi(x) = \frac{1}{D^3 + 8} (x^4 + 2x + 1)$$

$$= \frac{1}{8 \left(\frac{D^3}{8} + 1 \right)} x (x^4 + 2x + 1)$$

$$= \frac{1}{8 \left(1 + \frac{D^3}{8} \right)} x (x^4 + 2x + 1)$$

$$= \frac{1}{8} x \left(1 + \frac{D^3}{8} \right)^{-1} (x^4 + 2x + 1)$$

$$= \frac{1}{8} \left(1 - \frac{D^3}{8} + \frac{D^6}{64} - \dots \right) (x^4 + 2x + 1)$$

$$= \frac{1}{8} \left(x^4 + 2x + 1 - \frac{1}{8} (24x + 0 + 0) + 0 \right)$$

$$= \frac{1}{8} (x^4 + 2x + 1 - 3x)$$

$$= \frac{1}{8} (x^4 - x + 1)$$

Comp. soln.

$$y = C.F + P.I.$$

Q. Solve, $(D^3 - D^2 - 6D)y = e^x + x^2$.

$$m^3 - m^2 - 6m = 0$$

$$m(m^2 - m - 6) = 0$$

$$m(m^2 - 3m + 2m - 6) = 0$$

$$m[m(m-3) + 2(m-3)] = 0$$

$$m = 0, -2, 3$$

$$C.F = C_1 e^{0 \cdot x} + C_2 e^{-2x} + C_3 e^{3x}$$

$$P.I, \frac{1}{D^3 - D^2 - 6D} \times (e^x + x^2)$$

$$= \frac{1}{D^3 - D^2 - 6D} \times e^{1 \cdot x + 0} + \frac{1}{D^3 - D^2 - 6D} \times x^2$$

$$= -\frac{1}{6} e^x + \frac{1}{-6 \left(\left(-\frac{D^2}{6} + \frac{D}{6} \right) + 1 \right)} \times x^2$$

$$= -\frac{1}{6} e^x + \frac{x^2}{6D \left(1 + \left(-\frac{D^2}{6} + \frac{D}{6} \right) \right)}$$

$$= -\frac{1}{6} e^x + \frac{x^2}{6D} \times \left(1 + \left(-\frac{D^2}{6} + \frac{D}{6} \right) \right)^{-1}$$

$$= -\frac{1}{6} e^x + \frac{1}{6D} \times x^2 \left[1 + \frac{D^2}{6} - \frac{D}{6} + \left(\frac{D}{6} - \frac{D^2}{6} \right)^2 + \dots \right]$$

$$= -\frac{1}{6} e^x + \frac{1}{-6D} \left[x^2 + 2x + \frac{2}{3} \right]$$

$$= -\frac{1}{6} e^x - \frac{1}{6D} \left[x^2 - \frac{2x}{6} + \frac{2}{6} + \frac{2}{36} + 0 + 0 \right]$$

$$= -\frac{1}{6}e^x - \frac{1}{6} \left[\frac{x^3}{3} - \frac{2}{6}x \frac{x^2}{2} + \frac{2}{6}x + \frac{2}{36} \right]$$

* Case - IV $\rightarrow$ when $\phi(x) = e^{ax} V$, where V is a funcⁿ of x .

$$P.I., \frac{1}{F(D)} \times e^{ax} \times V = e^{ax} \times \frac{1}{f(D+a)} \times V$$

Q1

$$\text{Solve, } \frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = e^x \cos\left(\frac{x}{2}\right)$$

$$\text{For C.F., } D^2 y - 3Dy + 2y = 0$$

$$m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$m(m-2) - (m-2) = 0$$

$$m = 1, 2$$

$$C.F. = C_1 e^x + C_2 e^{2x}$$

$$P.I., \frac{1}{D^2 - 3D + 2} \times e^x \cos \frac{x}{2}$$

$$= e^x \times \frac{1}{(D+1)^2 - 3(D+1) + 2} \times \cos\left(\frac{x}{2}\right)$$

$$= e^x \times \frac{1}{D^2 + 1 + 2D - 3D - 3 + 2} \times \cos\left(\frac{x}{2}\right)$$

$$= e^x \times \frac{1}{D^2 - D} \times \cos \frac{x}{2}$$

$$= e^x \times \frac{1}{-\frac{1}{4} - D} \times \frac{\left(-\frac{1}{4} + D\right)}{\left(-\frac{1}{4} + D\right)} \times \cos\left(\frac{x}{2}\right)$$

$$= e^x \times \frac{D - \frac{1}{4}}{\frac{1}{16} - D^2} \times \cos\left(\frac{x}{2}\right)$$

$$= e^x \times \frac{D - \frac{1}{4}}{\frac{1}{16} + \frac{1}{4}} \times \cos\left(\frac{x}{2}\right)$$

$$= e^x \times \frac{(-\frac{1}{4} + D)}{\frac{5}{16}} \times \cos\left(\frac{x}{2}\right) = e^x \times \frac{-\cos(x/2) + D \cos(x/2)}{\frac{5}{16}}$$

$$= e^x \times \frac{-\cos(x/2)}{5} - \frac{1}{2} \frac{\sin x}{2}$$

$$PI = \frac{16}{5} e^x \left[\frac{-\cos(x/2)}{5} - \frac{\sin x/2}{2} \right]$$

$$C.F. y = PI + CF$$

Q. Solve, $\frac{d^2 y}{dx^2} - 4y = x \sinh x$.

C.F, $D^2 y - 4y = 0$
 $m^2 - 4 = 0$

$$(m-2)(m+2)$$

$$= 2, -2$$

$$C.F = c_1 e^{2x} + c_2 e^{-2x}$$

$$PI, \frac{1}{D^2 - 4} \times x \sinh x = \frac{1}{D^2 - 4} \times x \left(\frac{e^x - e^{-x}}{2} \right)$$

$$= \frac{1}{2} \left[\frac{1}{(D^2 - 4)} e^{1 \cdot x} - \frac{1}{(D^2 - 4)} x e^{-x} \right]$$

$$\frac{1}{2} \left[e^x \times \frac{1}{(D+1)^2 - 4} \times x - e^{-x} \times \frac{1}{(D-1)^2 - 4} \times x \right]$$

$$= \frac{1}{2} \left[e^x \times \frac{1}{D^2 + 1 + 2D - 4} \times x - e^{-x} \times \frac{1}{D^2 + 1 - 2D - 4} \times x \right]$$

$$= \frac{1}{2} \left[e^x \times \frac{1}{-3 \left(\frac{D^2}{-3} + \frac{2D}{-3} + 1 \right)} - e^{-x} \times \frac{1}{-3 \left(\frac{D^2}{-3} - \frac{2D}{-3} + 1 \right)} \times x \right]$$

$$= \frac{1}{2} \left[e^x \times \frac{1}{-3 \left(\frac{D^2}{-3} + \frac{2D}{-3} + 1 \right)} \times x - \frac{e^{-x}}{-3} \times \frac{1}{\left(\frac{D^2}{-3} - \frac{2D}{-3} + 1 \right)} \times x \right]$$

$$= \frac{1}{2} \left[e^x \times \frac{1}{-3 \left[1 - \left(\frac{D^2}{3} + \frac{2D}{3} \right) \right]} + \frac{e^{-x}}{3} \times \frac{1}{\left[1 - \left(\frac{D^2}{3} - \frac{2D}{3} \right) \right]} \times x \right]$$

$$= \frac{1}{2} \left[e^x \times \frac{1}{3 \left[1 - \left(\frac{D^2}{3} + \frac{2D}{3} \right) \right]} + \frac{e^{-x}}{3} \times \frac{1}{\left[1 - \left(\frac{D^2}{3} - \frac{2D}{3} \right) \right]} \times x \right]$$

$$= \frac{1}{2} \left[e^x \times \frac{1}{3 \left(1 + \left(\frac{D^2}{3} + \frac{2D}{3} \right) \right)} + \frac{e^{-x}}{3} \times \frac{1}{\left(1 + \left(\frac{D^2}{3} - \frac{2D}{3} \right) \right)} \times x \right]$$

Q. solve $(D^2 - 2D + 1)y = x e^x \cos x$

$$(m^2 - 2m + 1) = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$C.F = (C_1 + C_2 x) e^x$$

$$P.I = \frac{1}{D^2 - 2D + 1} (x e^x \cos x)$$

$$= e^x \times \frac{1}{D^2 - 2D + 1} (x \cos x)$$

$$= e^x \times \frac{1}{(D+1)^2 - 2(D+1) + 1} x \cos x$$

$$= e^x \times \frac{1}{D^2 + 1 + 2D - 2D - 2 + 1} x \cos x = \frac{e^x (x \cos x)}{D^2}$$

$$= e^x \times \frac{1}{D^2} x \cos x$$

$$= e^x \times \int \int \frac{x \cos x}{\pm \frac{D}{\mp}} dx = e^x \left[x \sin x - \int 1 \cdot \sin x dx \right]$$

$$= e^x \left[x \sin x + \cos x \right] = e^x \left[\int x \cos x + \int \cos x \right]$$

$$= e^x \left[-x \cdot \cos x + \int \cos x dx + \sin x \right]$$

$$= e^x \left[-x \cos x + \sin x + \sin x \right]$$

$$P.T = e^x \left[-x \cos x + 2 \sin x \right]$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Comp. solnⁿ, $y = CF + PI$

$$y = (c_1 + c_2 x) e^x + e^x [-x \cos x + 2 \sin x]$$

Case-5 $\rightarrow$ when $Q(x) = x^n \cos ax$ OR $x^n \sin ax$

$$P.I = \frac{1}{F(D)} x^n \cos ax = \text{Real part of } \frac{1}{F(D)} (x^n (\cos ax + i \sin ax))$$

$$= \text{Real part of } \frac{1}{F(D)} x^n e^{iax}$$

$$= \text{Real part of } \frac{1}{F(D)} e^{iax} \times x^n$$

$$\frac{1}{F(D)} x^n \cos ax = \text{Real part of } e^{iax} \times \frac{1}{F(D+ia)} x^n$$

similarly,

$$\frac{1}{F(D)} x^n \sin ax = \text{Imaginary part of } e^{iax} \times \frac{1}{F(D+ia)} x^n$$

Q. solve, $(D^2 - 2D + 1)y = x \sin x$

CF, $m^2 - 2m + 1 = 0$

$$m = 1, 1$$

$$CF = (c_1 + c_2 x) e^x$$

$$PI = \frac{1}{D^2 - 2D + 1} x \sin x = \frac{1}{D^2 - 2D + 1} x (\cos x + i \sin x)$$

$$= \text{Imag. part of } \frac{1}{D^2 - 2D + 1} x x e^{ix}$$

$$= \text{I.P. of } e^{ix} \times \frac{1}{(D+i)^2 - 2(D+i) + 1} x x$$

$$x + iy$$

$$= \text{I.P. of } e^{ix} \times \frac{1}{D^2 + i^2 + 2Di - 2D - 2i + 1} \times x$$

$$= \text{I.P. of } e^{ix} \times \frac{1}{D^2 + (i-1)2D - 2i} \times x$$

$$= \text{I.P. of } e^{ix} \times \frac{1 \times x}{-2i \left[\frac{D^2}{-2i} + \frac{(i-1)2D}{-2i} + 1 \right]}$$

$$= \text{I.P. of } e^{ix} \times \frac{1}{-2i} \left[1 - \left(\frac{D^2}{2i} + \frac{(i-1)D}{i} \right) \right] x$$

$$= \text{I.P. of } e^{ix} \times \frac{x}{2i} \left[1 + \frac{D^2}{2i} + \frac{(i-1)D}{i} \right]$$

$$\text{I.P. of } e^{ix} \times \frac{1}{2i} \left[x + 0 + \frac{i-1}{i} \right]$$

$$= \text{I.P. of } (\cos x + i \sin x) \times \frac{1}{2i} \left[\frac{ix + i - 1}{i} \right]$$

$$= \text{I.P. of } (\cos x + i \sin x) \times \frac{i(x+1) - 1}{-2i^2}$$

$$= \text{I.P. of } (\cos x + i \sin x) \times \frac{i(x+1) - 1}{2}$$

$$\text{I.P. of } \left[\frac{i(x+1)\cos x - \cos x + i^2 \sin(x+1) - i \sin x}{2} \right]$$

$$\text{P.I.} = (x+1)\cos x - \sin x$$

$$\frac{1}{(x+a)(x-a)} = \frac{1}{2a} \left[\frac{1}{x-a} - \frac{1}{x+a} \right]$$

General method to find P.I. of any function $\phi(x)$.
In this method, we resolve $F(D)$ into linear factors
of the form

$$(D-\alpha_1)(D-\alpha_2)(D-\alpha_3)\dots$$

corresponding to a factor, P.I. is given by,

$$\frac{1}{D-\alpha} \cdot \phi(x) = e^{\alpha x} \int e^{-\alpha x} \phi(x) dx$$

Q.

Solve the differential eqⁿ.

$$\frac{d^2 y}{dx^2} + a^2 y = \sec ax$$

$$D = \frac{d}{dx}$$

$$D^2 y + a^2 y = \sec ax$$

$$A.E. = m^2 + a^2 = 0$$

$$m^2 + a^2 = 0$$

$$m = \pm ai$$

$$m = 0 \pm ia$$

$$C.F. = e^{0 \cdot x} [C_1 \cos ax + C_2 \sin ax]$$

$$(1) \rightarrow e^{ax+1}$$

$$(2) \rightarrow \sin(ax+1) \text{ or } \cos(ax+1)$$

$$(3) \rightarrow x^m$$

$$(4) \rightarrow e^{ax} x^m$$

$$(5) \rightarrow \frac{1}{m} \cos ax \text{ or } \frac{1}{m} \sin ax$$

P.I.

$$\frac{1}{F(D)} \cdot \phi(x) = \frac{1}{D^2 + a^2} \cdot \sec ax = \frac{1}{D^2 - i^2 a^2} \cdot \sec ax$$

$$= \frac{1}{(D-ia)(D+ia)} \cdot \sec ax$$

$$= \frac{1}{2ia} \left[\frac{1}{D-ia} - \frac{1}{D+ia} \right] \cdot \sec ax$$

$$= \frac{1}{2ia} \left[\frac{1}{D-ia} \sec ax - \frac{1}{D+ia} \sec ax \right]$$

$$P.I. = \frac{1}{2ia} \left[\frac{1}{D-ia} \sec ax - \frac{1}{D+ia} \sec ax \right]$$

$$\left. \begin{aligned} e^{i\theta} &= \cos \theta + i \sin \theta \\ e^{-i\theta} &= \cos \theta - i \sin \theta \end{aligned} \right\}$$

$$P.I = \frac{1}{2ia} [P_1 - P_2] \quad \text{--- (I)}$$

$$P_1 = \frac{1}{D-ia} \sec ax = e^{iax} \int e^{-iax} \sec ax \, dx$$

$$= e^{iax} \int (\cos ax - i \sin ax) \times \frac{1}{\cos ax} \, dx$$

$$= e^{iax} \int (1 - i \tan ax) \, dx$$

$$P_1 = e^{iax} \left[x - i \frac{\log \sec ax}{a} \right] \quad \text{--- (II)}$$

Similarly, for P_2 replacing i by $-i$

$$P_2 = e^{-iax} \left[x + i \frac{\log \sec ax}{a} \right] \quad \text{--- (III)}$$

Putting in (I)

$$P.I = \frac{1}{2ia} \left[e^{iax} \left[x - i \frac{\log \sec ax}{a} \right] - e^{-iax} \left[x + i \frac{\log \sec ax}{a} \right] \right]$$

$$= \frac{1}{2ia} \left[e^{iax} x - i e^{iax} \frac{\log \sec ax}{a} - e^{-iax} x - i e^{-iax} \frac{\log \sec ax}{a} \right]$$

$$= \frac{1}{2ia} \left[x(e^{iax} - e^{-iax}) - i \frac{\log \sec ax}{a} [e^{iax} + e^{-iax}] \right]$$

$$= \frac{1}{2ia} \left[(e^{iax} - e^{-iax}) x - i \frac{\log \sec ax}{a} (e^{iax} + e^{-iax}) \right]$$

$$P.I = \frac{1}{2ia} \left[x \times 2i \sin ax - i \frac{\log \sec ax}{a} \times 2 \cos ax \right]$$

$$P.I = \frac{2i}{2ia} \left[x \sin ax - \frac{\log \sec ax}{a} \cos ax \right]$$

Comp. soln, $y = CF + P.I$

Q. Solve, $\frac{d^2 y}{dx^2} + \frac{3dy}{dx} + 2y = e^x$

$$A.E. = D^2 + 3D + 2 = 0$$

$$m^2 + 3m + 2 = 0$$

$$m^2 + 2m + m + 2 = 0$$

$$m(m+2) + 1(m+2) = 0$$

$$m = -1, -2$$

$$A.F = C_1 e^{-x} + C_2 e^{-2x}$$

$$P.I = \frac{1}{D^2 + 3D + 2} \cdot e^x = \frac{1}{(D+1)(D+2)} e^x$$

$$\frac{1}{(D+1)(D+2)} = \frac{A}{D+1} + \frac{B}{D+2}$$

$$1 = (D+1)^A + (D+2)^B$$

$$D(B+A) = 0$$

$$-A = -B$$

$$A + 2B = 1$$

$$-B + 2B = 1$$

$$B = 1$$

$$D+1=0$$

$$D = -1, D = -2$$

$$(D+2)B = 1$$

$$B = 1$$

$$P.I = \left(\frac{1}{D+1} - \frac{1}{D+2} \right) e^x$$

$$P.I = \frac{e^x}{D+1} - \frac{e^x}{D+2}$$

$$P.I = P_1 - P_2$$

$$\text{Now, } P_1 = \frac{e^x}{D+1} = \frac{1}{D-(-1)} \cdot e^x$$

$$= e^{-x} \int e^x e^x dx$$

$$\text{Let, } e^x = t, e^x dx = dt$$

$$= e^{-x} \int e^t dt = e^{-x} e^t$$

$$P_1 = e^{-x} e^{e^x}$$

$$P_2 = \frac{1}{D+2} e^{e^x} = \frac{1}{D-(-2)} e^{e^x}$$

$$= e^{-2x} \int e^{2x} \cdot e^{e^x} dx$$

$$= e^{-2x} \int e^x \cdot e^x \cdot e^{e^x} dx$$

$$e^x = t \Rightarrow e^x dx = dt$$

$$= e^{-2x} \int t e^t dt = e^{-2x} [t e^t - e^t]$$

$$P_2 = e^{-2x} [e^x e^{e^x} - e^{e^x}]$$

Putting in (1)

$$P.I. = e^{-x} e^{e^x} - e^{-2x} [e^x e^{e^x} - e^{e^x}]$$

$$P.I. = e^{-2x} e^{e^x}$$

Comp. solution

$$y = C.F. + P.I.$$

Q-1 Solve, $\frac{d^2 y}{dx^2} + y = \tan x$

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$C.F. = e^{0 \cdot x} [C_1 \cos x + C_2 \sin x]$$

$$P.I. = \frac{1}{D^2 + 1} \tan x = \frac{1}{D^2 - (-1)^2} \tan x = \frac{1}{(D-i)(D+i)} \tan x$$

$$\frac{1}{2i} \left[\frac{i}{D-i} - \frac{1}{D+i} \right] \cdot \tan x$$

$$= \frac{1}{2i} \left[\frac{1}{D-i} \tan x - \frac{1}{D+i} \tan x \right]$$

$$= \frac{1}{2i} \left[e^{ix} \int e^{-ix} \tan x - \right]$$

$$P \cdot I = \frac{1}{2i} [P_1 - P_2]$$

$$P_1 = \frac{1}{D-i} \tan x = e^{ix} \int e^{-ix} \tan x \, dx$$

$$P_1 = e^{ix} \int (\cos x - i \sin x) \tan x \, dx$$

$$= e^{ix} \int \left(\sin x - i \frac{\sin^2 x}{\cos x} \right) dx$$

$$= e^{ix} \int \sin x \, dx - i \int \frac{(\cos^2 x + 1)}{\cos x} \, dx$$

$$= e^{ix} \left[-\cos x + i \sin x + i \log(\sec x + \tan x) \right]$$

$$= e^{ix} [-\cos x + i \sin x + i \log(\sec x + \tan x)]$$

$$P_2 = e^{-ix} [-\cos x - i \sin x - i \log(\sec x + \tan x)]$$

$$P \cdot I = \frac{1}{2i} [P_1 - P_2]$$

$$= \frac{1}{2i} \left[e^{ix} [-\cos x + i \sin x + i \log(\sec x + \tan x)] - e^{-ix} [-\cos x - i \sin x - i \log(\sec x + \tan x)] \right]$$

$$= \frac{1}{2i} \left[-\cos x [e^{ix} - e^{-ix}] + i \sin x \right]$$

Euler-Cauchy Linear Differential Equation.

An eqⁿ of the form,

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = f(x)$$

where $a_0, a_1, a_2, \dots, a_n$ are all constant. $\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 7y = f(x)$

is called Euler-Cauchy Linear Differential Equation.

We substitute,

$$x = e^z \Rightarrow z = \log x$$

Now, $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \frac{dy}{dz} \times \frac{1}{x}$

$$x \frac{dy}{dx} = \frac{dy}{dz}$$

$$\left[x \frac{dy}{dx} = Dy \right], \text{ where } D = \frac{d}{dz}$$

Now,

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{dy}{dz} \times \frac{1}{x} \right)$$

$$= \frac{dy}{dz} \times \frac{-1}{x^2} + \frac{1}{x} \times \frac{d^2 y}{dz^2} \times \frac{dz}{dx}$$

$$\frac{d^2 y}{dx^2} = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x^2} \frac{d^2 y}{dz^2}$$

$$x^2 \frac{d^2 y}{dx^2} = -\frac{dy}{dz} + \frac{d^2 y}{dz^2}$$

$$x^2 \frac{d^2 y}{dx^2} = -Dy + D^2 y$$

$$\boxed{x^2 \frac{d^2 y}{dx^2} = D(D-1)y}$$

Similarly,

$$\boxed{x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y}$$

and so on.

Q. Solve, $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x + \log x$.

Given diff. eqⁿ is Euler-Cauchy linear differential eqⁿ.
 To solve, let $x = e^z \Rightarrow z = \log x$.

$$x \frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

Putting in given diff. eqⁿ

$$D(D-1)(D-2)y + 3D(D-1)y + Dy + y = e^z + z$$

$$= [D(D^2 - 3D + 2) + 3D^2 - 3D + D + 1]y = 0$$

$$\Rightarrow [D^3 - 3D^2 + 2D + 3D^2 - 2D + 1]y = e^z + z$$

$$\Rightarrow [D^3 + 1]y = e^z + z$$

$$A.E. \Rightarrow D \rightarrow m, y \rightarrow 1$$

$$m^3 + 1 = 0$$

$$(m+1)(m^2 - m + 1) = 0$$

$$m = -1, m = \frac{1 \pm \sqrt{1-4}}{2}$$

$$m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$C.F. = c_1 e^{-z} + e^{\frac{1}{2}z} \left[c_2 \cos \frac{\sqrt{3}}{2} z + c_3 \sin \frac{\sqrt{3}}{2} z \right]$$

$$P.I. = \frac{1}{D^3+1} (e^z + z)$$

$$= \frac{1}{D^3+1} \cdot e^z + \frac{1}{D^3+1} \cdot z$$

$$= \frac{1}{2} e^z + [1 + D^3]^{-1} z$$

$$= \frac{1}{2} e^z + [1 - D^3 + \dots] z$$

$$P.I. = \frac{1}{2} e^z + z$$

$$y = C.F. + P.I. = c_1 e^{-z} + e^{\frac{1}{2}z} \left[c_2 \cos \frac{\sqrt{3}}{2} z + c_3 \sin \frac{\sqrt{3}}{2} z \right] + \frac{1}{2} e^z + z$$

Replacing $z = \log x$.

$$y = c_1 e^{-\log x} + e^{\frac{1}{2} \log x} \left[c_2 \cos \frac{\sqrt{3}}{2} (\log x) + c_3 \sin \frac{\sqrt{3}}{2} (\log x) \right] + \frac{1}{2} e^{\log x} + \log x$$

Q. Solve, $x^4 \frac{d^3 y}{dx^3} + 2x^3 \frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$

$$x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \frac{1}{x}$$

Given diff. eqⁿ is Euler - Cauchy linear differential eqⁿ.

To solve, Let, $x = e^z \Rightarrow z = \log x$
 $x \frac{dy}{dx} = \frac{dy}{dz}$

$$x^2 \frac{d^2 y}{dx^2} = 3(D-1)y$$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

Putting in given diff. eqⁿ (1)

$$D(D-1)(D-2)y + 2D(D-1)y - Dy + y = \frac{1}{e^z}$$

$$[D^3 - 3D^2 + 2D + 2D^2 - 3D + 1]y = e^{-z}$$

$$= [D^3 - D^2 - D + 1]y = e^{-z}$$

A.E, $D \rightarrow m$ $y \rightarrow 1$

$$m^3 - m^2 - m + 1 = 0$$

$$m = 1, 1, -1$$

$$C.F = (c_1 + c_2 z)e^z + c_3 e^{-z}$$

$$P.I = \frac{1}{F(D)} \cdot \phi(z) = \frac{1}{D^3 - D^2 - D + 1} \times e^{-z}$$

$$= 2x \frac{1}{3D^2 - 2D - 1} \times e^{-z}$$

$$= 2x \frac{1}{3+2-1} \times e^{-z} = \frac{ze^{-z}}{4}$$

$$P.I = \frac{ze^{-z}}{4}$$

Comp. soluⁿ

$$y = C.F + P.I$$

$$y = (c_1 + c_2 z)e^z + c_3 e^{-z} + \frac{ze^{-z}}{4}$$

Q.

Solve

$$\left(\frac{d^2}{dx^2} + \frac{1}{x} \right) y = x^{-4}$$

$$\frac{d^2}{dx^2} + \frac{1}{x^2} + \left(2x \frac{d}{dx} \times \frac{1}{x} \right) y = x^{-4}$$

$$\left(\frac{d^2}{dx^2} + \frac{1}{x^2} + \frac{2}{x} \frac{d}{dx} \right) y = x^{-4}$$

$$\frac{d^2 y}{dx^2} + \frac{y}{x^2} + \frac{2}{x} \frac{dy}{dx} = x^{-4}$$

Multiply by x^2

$$x^2 \frac{d^2 y}{dx^2} + y + 2x \frac{dy}{dx} = x^{-2}$$

which is Euler - Cauchy diff. eqn

$$\text{let } x = e^z$$

$$x^2 \frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

Putting in given diff. eqn

$$D(D-1)y + y + 2Dy = e^{-2z}$$

$$m(m-1)y + 1 + 2m = 0$$

$$m^2 - m + 1 + 2m = 0$$

$$m^2 + m + 1 = 0$$

$$D = \frac{-1 \pm \sqrt{1-4}}{2}$$

Legendre's linear diff. eqⁿ

An equation of the form

$$a_0(ax+b)^n \frac{d^ny}{dx^n} + a_1(ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2(ax+b)^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_{n-1}(ax+b) \frac{dy}{dx} + a_n y = \phi(x)$$

is called Legendre's linear differential eqⁿ

let, $ax+b = e^z \Rightarrow z = \log(ax+b)$

then, $(ax+b) \frac{dy}{dx} = a \frac{dy}{dz}$

$$(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y$$

$$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$$

also on.

where,
 $D = \frac{d}{dz}$

Q. Solve the diff. eqⁿ

$$(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2) \frac{dy}{dx} = 36y - 3x^2 + 4x + 1$$

let, $3x+2 = e^z$

$z = \log(3x+2)$

$$(3x+2) \frac{dy}{dx} = 3 \frac{dy}{dz}$$

$$(3x+2)^2 \frac{d^2y}{dx^2} = 9 D(D-1)y$$

$$(3x+2)^3 \frac{d^3 y}{dx^3} = 27D(D-1)(D-2)y.$$

$$9D(D-1)y + 9Dy - 36 = 3x^2 + 4x + 1$$

$$9y(D(D-1) + D - 4) = 0.$$

$$D \rightarrow m, y \rightarrow 1.$$

$$9(m(m-1) + m - 4) = 0$$

$$m^2 - m + m - 4 = 0$$

$$(m-2)(m+2) = 0$$

$$m = 2, -2.$$

$$9D^2y - 9Dy + 9Dy - 36y = 3x^2 + 4x + 1.$$

$$9D^2y - 36y = 3\left(\frac{e^{2x}-2}{3}\right)^2 + 4\left(\frac{e^{2x}-2}{3}\right) + 1.$$

$$9D^2y - 36y = 3\left(\frac{e^{2x}+4-4e^{2x}}{3}\right) + 4\left(\frac{e^{2x}-2}{3}\right) + 1$$

$$9D^2y - 36y = \frac{e^{2x}+4-4e^{2x}+4e^{2x}-8}{3} + 1$$

$$9D^2y - 36y = \frac{e^{2x}-4}{3} + 1$$

$$9D^2y - 36y = \frac{e^{2x}-1}{3}$$

$$D \rightarrow m, y \rightarrow 1.$$

$$9m^2 - 36 = 0$$

$$m^2 - 4 = 0$$

$$m = 2, -2$$

$$C.F = c_1 e^{2x} + c_2 e^{-2x}$$

$$P.I = \frac{1}{9D^2 - 36} \left(\frac{e^{2z} - 1}{3} \right)$$

$$= \frac{1}{3} \times \frac{1}{(9D^2 - 36)} \times e^{2z} - \frac{1}{3} \times \frac{1}{9D^2 - 36} \times e^{0 \cdot z}$$

$$= \frac{1}{3} \times \frac{z}{(18D)} \times e^{2z} + \frac{1}{3} \times \frac{1}{36}$$

$$= \frac{1}{3} \times \frac{z \times e^{2z}}{36} + \frac{1}{3 \times 36} = \frac{z e^{2z} + 1}{3 \times 36}$$

$$y = C.F + P.I.$$

$$= C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{3} \left(\frac{z e^{2z} + 1}{36} \right)$$

Replacing, $z = \log(3x+2)$

$$y = C_1 e^{2 \log(3x+2)} + C_2 e^{-2 \log(3x+2)} + \frac{1}{3} \left(\frac{\log(3x+2) e^{2 \log(3x+2)} + 1}{36} \right)$$

Simultaneous Linear Diff. Equation

Q.

Solve, $\frac{dx}{dt} + x - 2y = 0$. Given $x(0) = y(0) = 1$

$$\frac{dy}{dt} + x + 4y = 0, \quad \frac{d}{dt} = D$$

$$Dx + x - 2y = 0 \Rightarrow (D+1)x - 2y = 0 \quad - (1)$$

$$Dy + x + 4y = 0 \Rightarrow (\cancel{D+4})y + x + (D+4)y = 0 \quad - (2)$$

Operating ① by $(D+4)$ and multiply ② by ②:

$$(D+4)(D+1)x - 2(D+4)y = 0$$

$$2x + 2(D+4)y = 0$$

$$(D+4)(D+1)x + 2x = 0$$

$$(D^2 + 5D + 6)x = 0$$

A.E., $D \rightarrow m, x \rightarrow 1$

$$m^2 + 5m + 6 = 0$$

$$m^2 + 3m + 2m + 6 = 0$$

$$m(m+3) + 2(m+3) = 0$$

$$m = -2, -3$$

$$C.F. = C_1 e^{-2t} + C_2 e^{-3t}$$

$$P.I. = 0$$

Comp. soln,

$$x = C.F. + P.I.$$

$$x = C_1 e^{-2t} + C_2 e^{-3t} \quad \text{--- (3)}$$

Putting in eq ①

$$(D+1)(C_1 e^{-2t} + C_2 e^{-3t}) - 2y = 0$$

$$-2C_1 e^{-2t} - 3C_2 e^{-3t} + C_1 e^{-2t} + C_2 e^{-3t} = 2y$$

$$y = -\frac{C_1}{2} e^{-2t} - C_2 e^{-3t} \quad \text{--- (4)}$$

given $x(0) = 1 \Rightarrow$ at $t=0, x=1$

and $y(0) = 1 \Rightarrow$ at $t=0, y=1$

From (3) & (4)

$$1 = C_1 + C_2$$

$$1 = -\frac{C_1}{2} - C_2$$

solving, $c_1 = 4$
 $c_2 = -3$

Putting in (3) & (4)

$$\begin{cases} x = 4e^{-2t} - 3e^{-3t} \\ y = -2e^{-2t} + 3e^{-3t} \end{cases}$$

Q.

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2 \cos t - 7 \sin t$$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4 \cos t - 3 \sin t$$

$$\text{Let } \frac{d}{dt} = D$$

$$Dx + Dy - 2y = 2 \cos t - 7 \sin t$$

$$Dx + (D-2)y = 2 \cos t - 7 \sin t \quad \text{--- (1) } \times D$$

$$(D+2)x - Dy = 4 \cos t - 3 \sin t \quad \text{--- (2) } \times (D-2)$$

$$D^2x + D(D-2)y = D(2 \cos t) - D(7 \sin t)$$

$$(D-2)(D^2+2)x - D(D-2)y = (D-2)(4 \cos t) - 3(D-2)(\sin t)$$

$$D^2x + (D-2)(D+2)x = -2 \sin t - 7 \cos t - 4 \sin t - 8 \cos t \\ - 3 \cos t + 6 \sin t$$

$$D^2x + (D-2)(D+2)x = -18 \cos t$$

$$(D^2 + D^2 - 4)x = -18 \cos t$$

$$\Rightarrow (2D^2 - 4)x = -18 \cos t$$

$$2m^2 - 4 = 0$$

$$m = \pm \sqrt{2}$$

$$m = 0 \pm \sqrt{2}$$

$$C.F = e^{0t} [C_1 \cosh \sqrt{2}t + C_2 \sinh \sqrt{2}t]$$

$$P.I = \frac{1}{2D^2 - 4} \times (-18 \cos t) = \frac{-18}{D^2 - 4} \cos t$$

$$P.I = 3 \cos t$$

comp. soluⁿ

$$x = C.F + P.I$$

$$x = C_1 \cosh \sqrt{2}t + C_2 \sinh \sqrt{2}t + 3 \cos t$$

putting in (2)

$$(D+2)(C_1 \cosh \sqrt{2}t + C_2 \sinh \sqrt{2}t + 3 \cos t) = 4 \cos t - 3 \sin t$$

$$-Dy = 4 \cos t - 3 \sin t$$

$$(D+2)(C_1 \cosh \sqrt{2}t + C_2 \sinh \sqrt{2}t + 3 \cos t) - Dy = 4 \cos t - 3 \sin t$$

Q. solve $\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$

$$\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t$$

$$\frac{d}{dt} = D$$

$$D^2x + Dy + 3x = e^{-t}$$

$$D^2y - 4Dx + 3y = \sin 2t$$

$$(D^2+3)x + Dy = e^{-t} \quad \text{--- (1)}$$

$$-4Dx + (D^2+3)y = \sin 2t \quad \text{--- (2)}$$

Operating (1) by (D^2+3) and (2) by D .

$$(D^2+3)^2x + D^2y = De$$

$$(D^2+3)^2x + D(D^2+3)y = (D^2+3)e^{-t}$$

$$-4D^2x + D(D^2+3)y = D\sin 2t$$

$$(D^2+3)^2x + 4D^2x = (D^2+3)e^{-t} - D\sin 2t$$

$$\Rightarrow [D^4 + 9 + 6D^2 + 4D^2]x = e^{-t} + 3e^{-t} - 2\cos 2t$$

$$\Rightarrow (D^4 + 10D^2 + 9)x = 4e^{-t} - 2\cos 2t$$

$$A.E, D \rightarrow m, x \rightarrow 1.$$

$$m^4 + 10m^2 + 9 = 0$$

$$(m^2+1)(m^2+9) = 0$$

$$m = \pm i, \pm 3i$$

$$m = 0 \pm i, 0 \pm 3i$$

$$C.F = e^{0t}(C_1 \cos t + C_2 \sin t) + e^{0t}(C_3 \cos 3t + C_4 \sin 3t)$$

$$P.I = P.I = \frac{1}{D^4 + 10D^2 + 9} (4e^{-t} - 2\cos 2t)$$

$$= 4 \times \frac{1}{D^4 + 10D^2 + 9} \times e^{-t} - 2 \times \frac{1}{D^4 + 10D^2 + 9} \times \cos 2t$$

$$= 4 \times \frac{1}{1+10+9} \times e^{-t} - 2 \times \frac{1}{(-4)^2 - 40 + 9} \times \cos 2t$$

$$= \frac{4}{20} e^{-t} + \frac{2}{15} \cos 2t$$

comp. soln:

$$x = CF + PT$$

$$x = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{4}{20} e^{-t} + \frac{2}{15} \cos 2t$$

Putting in Eq (1)

$$(D^2 + 3)(C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{4}{20} e^{-t} + \frac{2}{15} \cos 2t) + Dy = e^{-t}$$

$$Dy = e^{-t}$$

Integrating

To find comp. solution of $\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R$ by changing independent variable.

Consider the diff. eqn.

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R$$

where P, Q, R are functⁿ of x
from x to z .

To solve this diff. eqn, we change its independent variable from x to z & find solution as follows.

Step-I $\rightarrow$ compare the diff. eqⁿ with standard form.

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$$

I find P, Q, R .

Step-II $\rightarrow$ take z such that $\left(\frac{dz}{dx}\right)^2 = Q$

here Q is taken in such a way that it remains the perfect square of a function.

Step-III Find $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$ and z

Step-IV

now,

$$\text{Find } P_1 = \frac{P \frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Step V $\rightarrow$ substitute these values in $\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$, which will be the diff. eqⁿ with constant coeff. & can be solved by previous method.

Imp. note $\rightarrow$

- ① This method is applicable only when the value of Q is a perfect square in terms of variable x .

Q. Solve, $x^2 \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 8x^3 \sin x^2$.

Dividing by x .

$$\frac{d^2 y}{dx^2} - \frac{1}{x} \frac{dy}{dx} - 4x^2 y = 8x^2 \sin x^2$$

Comparing with $\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Q y = R$

$$P = -\frac{1}{x} \quad Q = -4x^2 \quad R = 8x^2 \sin x^2$$

$$\text{Let, } \left(\frac{dz}{dx} \right)^2 = 4x^2 \Rightarrow \frac{dz}{dx} = 2x$$

$$\text{Integral, } z = x^2 \text{ and } \frac{d^2 z}{dx^2} = 2$$

$$\text{now, } P_1 = P \frac{dz}{dx} + \frac{d^2 z}{dx^2} = \frac{1}{x} \times 2x + 2$$

$$\left(\frac{dz}{dx} \right)^2 = 4x^2$$

$$P_1 = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx} \right)^2} = \frac{-4x^2}{4x^2} = -1$$

$$Q_1 = -1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx} \right)^2} = \frac{8x^2 \sin x^2}{4x^2}$$

$$(R_1 = 2 \sin x^2) = 2 \sin z$$

$$\frac{d^2 y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

$$\frac{d^2 y}{dz^2} + 0 + (-1)y = 2 \sin z$$

$$\frac{d^2 y}{dz^2} - y = 2 \sin z$$

$$\text{let } \frac{d}{dz} = D$$

$$D^2 y - y = 2 \sin z$$

$$(D^2 - 1)y = 2 \sin z$$

$$\text{C.F. } D \rightarrow m, y \rightarrow 1.$$

$$(m^2 - 1)y = 0$$

$$m = \pm 1$$

$$\text{C.F.} = c_1 e^z + c_2 e^{-z}$$

$$P.I = \frac{1}{D^2 - 1} \times 2 \sin z = 2 \times \frac{1}{-1 - 1} \times \sin z$$

$$P.I = -\sin z$$

Comp. soluⁿ.

$$y = \text{C.F.} + P.I.$$

$$y = c_1 e^z + c_2 e^{-z} - \sin z$$

Replacing $z = x^2$

$$y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin(x^2)$$

Ans

9 To find complete solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy^2 = R$
by normal form method (Removal of first order
derivative)



Ref] Compare the differential eqⁿ with standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{and find } P, Q, R$$

II let $y = uv$ is complete solⁿ.

III find $u = e^{-\frac{1}{2}\int P dx}$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} \quad \text{and } S = \frac{R}{u}$$

IV Normal form eqⁿ is

$$\frac{d^2v}{dx^2} + Iv = S$$

Solve this eqⁿ for v

V $y = uv$ will be complete solⁿ.

⇒ This method is applicable only when the value of $I = \text{const or constant } 0 \text{ or const.}$
 $\frac{1}{x} \quad (ax+b)^2$

Q Solve the differential eqⁿ by normal form

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 - 8)y = x^2 e^{-x^2/2}$$

Solⁿ

$$P = 2x \quad Q = x^2 - 8 \quad R = x^2 e^{-x^2/2}$$

$$\text{and } u = e^{-\frac{1}{2}\int P dx}$$

$$= e^{-\frac{1}{2}\int 2x dx}$$

$$\Rightarrow \boxed{u = e^{-x^2/2}}$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$I = x^2 - 8 - \frac{1}{2} \times 2x - \frac{4x^2}{4}$$

$$\boxed{I = -9}$$

$$\text{and } S = \frac{R}{u} = \frac{x^2 e^{-x^2/2}}{e^{-x^2/2}}$$

So normal form is

$$\frac{d^2 V}{dx^2} + IV = S$$

$$\frac{d^2 V}{dx^2} - gV = x^2$$

$$\frac{d}{dx} = D$$

$$D^2 V - gV = x^2$$

$$(D^2 - g)V = x^2$$

$$AE \quad D \rightarrow m \quad x \rightarrow 0$$

$$m = -3, +3$$

$$CF = C_1 e^{-3x} + C_2 e^{3x}$$

$$PI = \frac{1}{D^2 - g} x^2$$

$$= \frac{1}{-g \left[\frac{D^2}{g} + 1 \right]} x^2$$

$$= -\frac{1}{g} \left[1 - \frac{D^2}{g} \right]^{-1} x^2$$

$$= -\frac{1}{g} \left[1 + \frac{D^2}{g} + \dots \right] x^2$$

$$PI = -\frac{1}{g} \left[x^2 + \frac{2}{g} \right]$$

Solⁿ

$$V = CF + PI$$

$$V = C_1 e^{-3x} + C_2 e^{3x} - \frac{1}{g} \left(x^2 + \frac{2}{g} \right)$$

Complete solⁿ



$$y = e^{-x^{1/2}} [C_1 e^{3x} + C_2 e^{-3x} - \frac{1}{9} (\frac{x^2+2}{9})]$$

$$\frac{d^2 y}{dx^2} + \frac{1}{x^{1/3}} \frac{dy}{dx} + \left(\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} \right) y = 0$$

$$P = \frac{1}{x^{1/3}} \quad Q = \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} \quad R = 0$$

$$u = e^{-\frac{1}{2} \int P dx} = e^{-\frac{1}{2} \int \frac{1}{x^{1/3}} dx}$$

$$= e^{\frac{3}{4} x^{2/3}}$$

$$\text{and } I = Q - \frac{1}{2} \frac{dP}{dx} - \frac{P^2}{4}$$

$$I = \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} - \frac{1}{2} x \cdot \frac{1}{3} e^{-4/3} - \frac{1}{4x^{1/2}}$$

$$I = \frac{1}{6x^{4/3}} - \frac{6}{x^2} + \frac{1}{6} x^{4/3}$$

$$= \frac{-6}{x^2}$$

$$R - S = 0 = \left(\frac{R}{u} \right)$$

$$\text{normal form is } \frac{d^2 v}{dx^2} + I v = 0$$

$$\frac{d^2 v}{dx^2} - \frac{6}{x^2} v = 0$$

Tutorial-1Q1 solve.

Find comp. solnⁿ of $\frac{d^2y}{dx^2} + p \frac{dy}{dx} + Qy = R$ by method of reduction of order.

Steps for solution

Step-1 compare $\frac{d^2y}{dx^2} + p \frac{dy}{dx} + Qy = R$.

Find p, Q, R .

Step-2 Let $y = uv$ is complete solution.

Step-3 u is the part of C.F. (one solu) of diff. eqⁿ given by following conditions.

Condition $\Rightarrow$ part of C.F.

① If $m^2 + mp + Q = 0 \rightarrow e^{mx}$
for $m = 1$

$1 + p + Q = 0 \rightarrow e^x$

for $m = -1$

$1 - p + Q = 0 \rightarrow e^{-x}$

② If $m(m-1) + mp + Q = 0 \rightarrow x^m$
for $m = 1$

$1x + Qx^2 = 0$

$\Rightarrow 1 + Qx = 0 \rightarrow x$

for $m = 2$.

$2 + 2px + Qx^2 = 0 \rightarrow x^2$

Step IV $\rightarrow$ v is given by $\frac{dz}{dx} + \left(P + \frac{2}{u} \frac{du}{dx} \right) z = \frac{R}{u}$, where $z = \frac{dy}{dx}$

Step V $\rightarrow$ complete solution will be $y = uv$.

Q. Solve the diff. eqⁿ,

$$x^2 \frac{d^2 y}{dx^2} - 2x(1+x) \frac{dy}{dx} + 2(1+x)y = x^3$$

by method of reduction of order

Dividing by x^2

$$\frac{d^2 y}{dx^2} - \frac{2}{x}(1+x) \frac{dy}{dx} + \frac{2}{x^2}(1+x)y = x$$

Comparing with $\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R$

$$P = -\frac{2}{x}(1+x), \quad Q = \frac{2}{x^2}(1+x), \quad R = x$$

let, $y = uv$ is comp solⁿ.

$$P + Qx = 0$$

So, part of C.F is x .

$$\text{i.e., } \boxed{u = x}$$

now, v is given by

$$\frac{dz}{dx} + \left(P + \frac{2}{u} \frac{du}{dx} \right) z = \frac{R}{u}, \quad \text{where } z = \frac{dv}{dx}$$

$$\frac{dz}{dx} + \left[-\frac{2}{x}(1+x) + \frac{2}{x} \times 1 \right] z = \frac{x}{x}$$

$$\frac{dz}{dx} + \left[-\frac{2}{x} - 2 + \frac{2}{x} \right] z = 1$$

$$\frac{dz}{dx} - 2z = 1$$

$$\frac{d}{dx} = D; \quad D z - 2z = 1$$

$$(D-2)z = 1$$

$$A.E, D \rightarrow m; y \rightarrow 1$$

$$m-2=0 \Rightarrow m=2$$

$$C.F = C_1 e^{2x}$$

$$P.I = \frac{1}{D-2} \cdot e^{0 \cdot x} = \frac{1}{0-2} e^{0 \cdot x} = -\frac{1}{2}$$

$$P.I = -\frac{1}{2}$$

$$\text{So, soln}^n, z = C.F + P.I$$

$$= C_1 e^{2x} - \frac{1}{2}$$

$$z = \frac{dv}{dx} = C_1 e^{2x} - \frac{1}{2}$$

$$dv = \left(C_1 e^{2x} - \frac{1}{2} \right) dx$$

Integrating,

$$v = \frac{C_1 e^{2x}}{2} - \frac{1}{2} x + C_2$$

comp. solnⁿ, $y = uv$

$$y = x \left[\frac{C_1 e^{2x}}{2} - \frac{1}{2} x + C_2 \right] \quad \text{Ans}$$

Q

Solve,

$$y'' - 4xy' + (4x^2 - 2)y = 0$$

Given $y = e^{x^2}$ is a solution.

$$u = e^{x^2}$$

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R.$$

$$P = -4x \quad Q = 4x^2 - 2 \quad R = 0$$

$$\frac{dz}{dx} + \left(P + \frac{z}{u} \frac{du}{dx} \right) = \frac{Ru}{u} \quad \text{where, } z = \frac{dv}{dx}$$

$$\frac{dz}{dx} + \left(-4x + \frac{2}{e^{x^2}} \times e^{x^2} \times 2x \right) z = 0$$

$$\frac{dz}{dx} + [-4x + 4x]z = 0$$

$$dz/dx = 0$$

$$\text{Integrating, } \boxed{z = C_1}$$

$$\frac{dv}{dx} = C_1$$

$$\boxed{v = C_1 x + C_2}$$

Comp. soln, $y = uv$

$$y = e^{x^2} [C_1 x + C_2]$$

Ans

Q. Solve,

$$y'' - \cot x y' - (1 - \cot x)y = e^x \sin x$$

$$P = -\cot x \quad Q = -(1 - \cot x) \quad R = e^x \sin x$$

$$\frac{-\cot x - 1 + \cot x + 1}{1 + P + Q} = 0$$

$$1 + P + Q = 0$$

$$(u = e^x)$$

$$\frac{dz}{dx} + \left(P + \frac{2}{u} \frac{du}{dx} \right) z = \frac{R}{u} \quad \text{where } z = \frac{dv}{dx}$$

$$\frac{dz}{dx} + \left[-\cot x + \frac{2}{e^x} \times e^x \right] z = \frac{e^x \sin x}{e^x}$$

$$\frac{dz}{dx} + (\cot x + 2)z = \sin x$$

$$P = -(\cot x + 2) \quad Q = \sin x$$

$$I.F = e^{\int P dx} = e^{-\int (\cot x + 2) dx} = e^{-\log(\sin x) - 2x} = \frac{1}{\sin x} \cdot e^{-2x}$$

$$y \times I.F = \int Q \cdot I.F dx$$

$$I.F = \frac{e^{-2x}}{\sin x}$$

$$y \times \frac{e^{-2x}}{\sin x} = \int \sin x \times \frac{e^{-2x}}{\sin x} dx$$

$$\frac{z e^{-2x}}{\sin x} = \frac{e^{-2x}}{2} + C$$

$$z = \frac{\sin x}{2} + \frac{C \sin x}{e^{2x}} \quad z = \frac{dv}{dx}$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$dv = z \, dx$$

Integrating,

$$v = \left[\frac{\sin x}{2} + \frac{C \sin x}{e^{2x}} \right] dx$$

$$v = -\frac{\cos x}{2} + C \int e^{-2x} \sin x \, dx$$

$$v = -\frac{\cos x}{2} + \frac{C_1 e^{-2x}}{5} (-2 \sin x - \cos x)$$

Method of Variation of Parameters Ans

Consider diff. eqⁿ

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Q y = R$$

Let, C.F. of this diff. eqⁿ is

$$C.F. = C_1 y_1 + C_2 y_2$$

then by variation of parameter

$$P.I. = u y_1 + v y_2$$

$$\text{where, } u = \int \frac{R y_2}{y_1 y_2' - y_1' y_2} \, dx$$

$$\& v = \int \frac{R y_1}{y_1 y_2' - y_1' y_2} \, dx$$

Q. Apply method of variation of parameters to solve the differential eqⁿ, $\frac{d^2y}{dx^2} + y = \tan x$.

Comparing the eqⁿ, $P = D$, $Q = 1$, $R = \tan x$.

Let, $\frac{d}{dx} = D$.

$$D^2y + y = \tan x.$$

$$A.E = m^2 + 1 = 0$$

$$m = \pm \sqrt{-1}$$

$$m = \pm i$$

$$C.F = e^{0x} [C_1 \cos x + C_2 \sin x]$$

$$= C_1 \cos x + C_2 \sin x.$$

$\therefore$ Comparing by $C.F = C_1 y_1 + C_2 y_2$.

$$\therefore y_1 = \cos x, \quad y_2 = \sin x$$

then by variation of parameters

$$P.I = uy_1 + vy_2$$

$$u = \int \frac{-\tan x \times \sin x}{\cos x \cos x + \sin x \times \sin x} dx$$

$$u = \int \frac{-\sin^2 x}{\cos x} dx.$$

$$u = - \int \tan x \sec x dx. \quad u = - \int \frac{1 - \cos^2 x}{\cos x} dx$$

$$u = - \int (\sec x - \cos x) dx = - \log |\sec x + \tan x| + \sin x$$

$$u = - \log |\sec x + \tan x| + \sin x$$

Now, $v = \int \frac{ly_1}{y_1 y_2^2} - y_1' y_2 dx$

$$= \int \frac{\tan x \times \cos x}{\boxed{v = -\cos x}} dx = \int \sin x dx$$

So, $PI = uy_1 + vy_2$

$$PI = -(\log(\sec x + \tan x) \cos x) + \cos x \times \sin x$$

$$PI = -(\log(\sec x + \tan x) \cos x) - \cos x \sin x$$

$$\boxed{PI = -\log(\sec x + \tan x) \cos x}$$

$$y = CF + PI$$

$$\boxed{y = C_1 \cos x + C_2 \sin x - \cos x \log(\sec x + \tan x)}$$

Q.

Solve by V.O.P.

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^3 \log x$$

which is Euler Cauchy diff. eqⁿ.

Let, $x = e^z$

$z = \log x$

$x \frac{dy}{dx} = Dy$

$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$

$p = \frac{2}{x}$

$q = \frac{-12}{x^2}$

$R = x \log x$

$$D(D-1)y + 2Dy - 12y = ze^{3z}$$

$$y[D^2 + D - 12] = ze^{3z}$$

$$A.E, \quad m^2 + m - 12 = 0$$

$$m^2 + 4m - 3m - 12 = 0$$

$$m(m+4) - 3(m+4) = 0$$

$$m = 3, -4$$

$$C.F = c_1 e^{-4z} + c_2 e^{3z}$$

$$\therefore y_1 = e^{-4z} \quad y_2 = e^{3z}$$

$$u = \int \frac{-Ry_2}{y_1 y_2' - y_1' y_2} dz = - \int \frac{e^{3z} \cdot z \cdot e^{3z}}{3e^{-4z} e^{3z} + 4e^{-4z} \times e^{3z}} dz$$

$$u = - \int \frac{z e^{6z}}{7e^{-z}} dz$$

$$u = -\frac{1}{7} \int z \cdot e^{7z} dz = -\frac{1}{7} \left[\frac{z \cdot e^{7z}}{7} - \int \frac{1 \cdot e^{7z}}{7} dz \right]$$

$$u = -\frac{1}{7} \left[\frac{z \cdot e^{7z}}{7} - \frac{1}{7} \frac{e^{7z}}{7} \right] = -\frac{e^{7z}}{49} \left[z - \frac{1}{7} \right]$$

$$u = -\frac{1}{7} \left[\frac{z e^{7z}}{7} - \frac{e^{7z}}{49} \right]$$

$$v = \int \frac{-Ry_1}{y_1 y_2' - y_1' y_2} dz = - \int \frac{z e^{3z} \times e^{-4z}}{7e^{-z}} dz = -\frac{1}{7} \int z dz$$

$$v = -\frac{z^2}{14}$$

$$PI = uy_1 + vy_2$$

$$P.I = -\frac{1}{7} \left[\frac{ze^{7z}}{7} - \frac{e^{7z}}{49} \right] x e^{-4z} + \frac{z^2}{14} x e^{3z}$$

Comp

$$PI = \left[\frac{-ze^{7z}}{49} + \frac{e^{7z}}{343} \right] e^{-4z} + \frac{z^2}{14} e^{3z}$$

$$PI = \left[\frac{-ze^{3z}}{49} + \frac{e^{3z}}{343} + \frac{z^2}{14} e^{3z} \right]$$

$$y = C.F + PI = C_1 e^{-4z} + C_2 z e^{3z} + \frac{e^{3z}}{343} + \frac{z^2}{14} e^{3z} - \frac{ze^{3z}}{49}$$

Replace, z by $\log x$

~~y =~~

Q.

Use VOP method to solve the differential eqⁿ

$$y'' - 2y' + y = e^x \log x$$

$$P = -2 \quad Q = 1 \quad R = e^x \log x$$

$$\text{let, } \frac{d}{dx} = D$$

$$D^2 y - 2Dy + y = e^x \log x$$

$$\text{A.E, } m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$C.F = (C_1 + C_2 x) e^x$$

$$C.F = C_1 e^x + C_2 x e^x$$

$$y_1 = e^x \quad y_2 = x e^x$$

$$PI = uy_1 + vy_2$$

$$u = \frac{\int Ry_2}{y_1 y_2' - y_1' y_2} = \frac{-e^x \log x \cdot x e^x}{e^x [1 \cdot e^x + e^x x] - e^x x e^x}$$

$$u = \frac{-\int x \log x e^{2x}}{e^{2x} + e^{2x} x - e^{2x} x}$$

$$u = - \int \frac{x^2 \log x}{e^{2x}} = - \int x \log x \cdot dx$$

Let, $\log x = t$
 $\frac{1}{x} dx = dt$

$$u = - \left[\log x \times \frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2} dx \right]$$

$$u = - \frac{x^2 \log x}{2} - \frac{x^2}{4}$$

$$u = - \frac{x^2}{2} \left[\log x + \frac{1}{2} \right]$$

$$v = \frac{-\int Ry_1}{y_1 y_2' - y_1' y_2} = - \int \frac{e^x \log x x e^x}{e^{2x}}$$

$$= - \int \log x \cdot 1 dx = - \left[\log x \times x - \int \frac{1}{x} \times x dx \right]$$

$$v = -x [\log x - 1]$$

$$PI = uy_1 + vy_2$$

$$= -\frac{x^2}{2} \left[\log x + \frac{1}{2} \right] x e^x - x [\log x - 1] x x e^x$$

$$y = C.F + PI$$

$$y = C_1 e^x + C_2 x e^x + \frac{x^2}{2} \left[\log x + \frac{1}{2} \right] x e^x - x [\log x - 1] x x e^x$$

Ans

Application of Ordinary Differential Equation

Application to Electric Circuit

Some Basic Results.

① The voltage drop across Inductance $(L) = L \frac{di}{dt}$

② Resistance $(R) = Ri$

③ Capacitance $(C) = \frac{q}{C}$

④ $i = \frac{dq}{dt}$

① $L-R$

② $L-C$

③ $R-C$

④ $L-R-C$

Kirchoff's Law

The total voltage drop in a ckt is equal to the resultant emf (applied voltage).

Q An inductance L of $5H$ & a resistance R of 25Ω are connected in series with an emf ' E ' Volt. If the current is zero, when time ' t ' = 0 , then find the current at the end of 1 second, if $E = 100V$

By Kirchoff's Law

$$L \frac{di}{dt} + Ri = E$$

$$5 \frac{di}{dt} + 25i = 100$$

$$\frac{di}{dt} + 5i = 20$$

$$\text{Let, } \frac{d}{dt} = D$$

$$Di + 5i = 20$$

$$(D+5)i = 20$$

$$\text{A.E, } m+5=0$$

$$m = -5$$

$$C.F = C_1 e^{-5t}$$

$$P.I = \frac{1}{D+5} \times 20 = \frac{20 \times 1}{D+5} e^{0 \cdot t} = 4$$

$$P.I = 4$$

$$i = C_1 e^{-5t} + 4$$

$$\text{at } t=0, i=0$$

$$C_1 = -4$$

$$\text{at } t=1$$

$$i = C_1 e^{-5} + 4 = -4e^{-5} + 4$$

$$i = 3.973$$

Q. Ckt. consists of inductance of $10H$, Resistance of 90Ω & a condenser of capacitance $\frac{1}{10} F$. If initially $q=0$ at $t=0$, & $i=1$ at $t=\frac{1}{10}$. Find the current & charge if there is an alternating emf $E = 10 \cos t$.

$$\text{By Kirchoff's law; } L \frac{di}{dt} + Ri + \frac{q}{C} = E$$

$$i = \frac{dq}{dt}$$

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E$$

$$10 \frac{d^2 q}{dt^2} + 90 \frac{dq}{dt} + 140q = 10 \cos t$$

$$\frac{d^2 q}{dt^2} + 9 \frac{dq}{dt} + 14q = \cos t$$

$$\text{let, } \frac{d}{dt} = D$$

$$(D^2 + 9D + 14)q = \cos t$$

$$\text{ie } D \rightarrow m, q \rightarrow I$$

$$m^2 + 9m + 14 = 0$$

$$m^2 + 2m + 7m + 14 = 0$$

$$m(m+2) + 7(m+2) = 0$$

$$m = -2, -7$$

$$C.F = C_1 e^{-2x} + C_2 e^{-7x}$$

$$P.I = \frac{1}{D^2 + 9D + 14} \times \cos t = \frac{1}{-9 + 0 - 1 + 9D + 14} \cos t$$

$$= \frac{1}{9D + 13} \times \frac{9D - 13}{9D - 13} \cos t = \frac{9D - 13}{81D^2 - 169} \cos t$$

$$= \frac{9D - 13}{1 - 81 - 169} \cos t = \frac{-9 \sin t - 13 \cos t}{-250}$$

$$P.I = \frac{9 \sin t + 13 \cos t}{250}$$

complete solution,

$$q = C.F + P.I$$

$$q = c_1 e^{-2t} + c_2 e^{-7t} + \frac{9 \sin t + 13 \cos t}{250} \quad (1)$$

$$i = \frac{dq}{dt} = -2c_1 e^{-2t} - 7c_2 e^{-7t} + \frac{9 \cos t - 13 \sin t}{250} \quad (2)$$

Given at $t=0$, $q=0$

at $t=0$, $i=1$

From eq (1) & (2)

$$0 = c_1 + c_2 + \frac{13}{250} \Rightarrow c_1 + c_2 = -\frac{13}{250} \quad (3)$$

$$1 = -2c_1 - 7c_2 + \frac{9}{250} \Rightarrow -2c_1 - 7c_2 = \frac{241}{250} \quad (4)$$

On solving

$$c_1 = 0.12, c_2 = -0.172$$

Putting in (1) & (2)

$$q = \checkmark$$

$$i = \checkmark$$

Ans

Q. A ckt. consists of an inductance 'L' & a condenser of capacity 'C' in series, an alternating emf $E \sin nt$ is applied to the ckt. at time $t=0$. The initial current and the charge on the condenser are 0. Prove that the current at any time 't' is

$$i = \frac{nE}{L(n^2 - \omega^2)} \times (\cos \omega t - \cos nt), \text{ where } \omega^2 = \frac{1}{LC}.$$

By Kirchhoff's Law,

$$L \frac{di}{dt} + \frac{q}{C} = E$$

$$L \frac{d^2 i}{dt^2} + \frac{q}{C} = E \sin nt$$

$$L \frac{d^2 q}{dt^2} + \frac{q}{C} = E \sin nt$$

$$\text{Let, } \frac{d^2 q}{dt^2} + \frac{1}{LC} \cdot q = \frac{E \sin nt}{L}$$

$$\frac{d^2 q}{dt^2} + \omega^2 q = \frac{E \sin nt}{L}$$

$$\text{Let, } \frac{d}{dt} = D$$

$$(D^2 + \omega^2) q = \frac{E \sin nt}{L}$$

$$A.E, \quad m^2 + \omega^2 = 0$$

$$(m = \pm \omega i)$$

$$C.F = e^{0 \cdot x} [C_1 \cos \omega t + C_2 \sin \omega t]$$

$$(C.F = C_1 \cos \omega t + C_2 \sin \omega t)$$

$$P.I = \frac{1}{D^2 + \omega^2} \times \frac{E \sin nt}{L}$$

$$= \frac{E}{L} \times \frac{1}{D^2 + \omega^2} \times \sin nt$$

$$P.I = \frac{E}{L} \times \frac{1}{-\omega^2 + \omega^2} \times \sin nt$$

$$q = C.F + P.I = C_1 \cos \omega t + C_2 \sin \omega t + \frac{E}{L} \times \frac{\sin nt}{\omega^2 - n^2}$$

$$i = \frac{dq}{dt} = \frac{d}{dt} (C_1 \cos \omega t + C_2 \sin \omega t + \frac{E}{L} \times \frac{\sin nt}{\omega^2 - n^2})$$

Q. An alternating emf, $E = 200 e^{-5t}$ is connected in series with a resistor of 20Ω & a capacitor of 0.01 Farad. Assuming $Q = 0$ at $t = 0$. Find the charge and the current at any time 't'.

By Kirchhoff's law,

$$Ri + \frac{q}{C} = E$$

$$[i = dq/dt]$$

$$R \frac{dq}{dt} + \frac{q}{C} = 200 e^{-5t}$$

$$20 \frac{dq}{dt} + \frac{q}{0.01} = 200 e^{-5t}$$

$$\text{Let } \frac{d}{dt} = D$$

$$20 Dq + \frac{q}{0.01} = 200 e^{-5t}$$

$$\left(20D + \frac{100}{0.01}\right) q = 200 e^{-5t}$$

$$D \rightarrow m, \quad q \rightarrow 1$$

$$A.E, \quad 20m + \frac{1}{0.01} = 0$$

$$20m = -\frac{1}{0.01}$$

$$m = -\frac{1}{0.01 \times 20}, \quad m = -5$$

$$[C.F = C_1 e^{-5t}]$$

$$P.I = \frac{1}{F(D)} \phi(q, t) = \frac{1}{20D + 100} \times 200 e^{-5t}$$

$$= 200 \times \frac{1}{20D + 100} \times e^{-5t} = -\frac{200}{99} \times e^{-5t}$$

$$[P.I = -\frac{200}{99} e^{-5t}]$$

$$q = CF + PI$$

$$q = c_1 e^{-5t} - \frac{200 \cdot e^{-5t}}{99}$$

$$q = c_1 e^{-5t}$$

$$PI = \frac{10}{20} \times \frac{1}{20} \times e^{-5t} \times t$$

$$PI = 10t e^{-5t}$$

$$q = CF + PI = c_1 e^{-5t} + 10t e^{-5t} = 0$$

at, $q = 0$, $t = 0$

$$c_1 e^0 = 0$$

$$c_1 = 0$$

$$\therefore q = 10t e^{-5t}$$

$$i = \frac{dq}{dt} = 10 \frac{d(t \cdot e^{-5t})}{dt}$$

$$= 10 [-5t e^{-5t} + e^{-5t}]$$

$$\bar{i} = 10 e^{-5t} [1 - 5t]$$

$$i = 10 e^{-5t} [1 - 5t]$$

Q. An R, L ckt has an emf $4 \sin t$, a resistance of 100Ω & an inductance of $4H$, with no initial current. Find the current at any time 't'.

$$L \frac{di}{dt} + Ri = E$$

$$4 \frac{di}{dt} + 100i = 4 \sin t$$

$$i = dq/dt$$

$$4 \frac{d^2 q}{dt^2} + 100 \frac{dq}{dt} = 4 \sin t$$

$$\text{Let, } \frac{d}{dt} = D$$

$$4D^2 q + 100Dq = 4 \sin t$$

$$(D^2 + 25D)q = \sin t$$

$$\text{A.E, } D \rightarrow m, q \rightarrow 1$$

$$m^2 + 25m = 0$$

$$m(m+25) = 0$$

$$m = 0, -25$$

$$\text{C.F} = C_1 e^{0 \cdot t} + C_2 e^{-25t} = C_1 + C_2 e^{-25t}$$

$$\boxed{\text{C.F} = C_1 + C_2 e^{-25t}}$$

$$P.I = \frac{1}{D^2 + 25D} \times \sin t = \frac{1}{-1 + 25D} \times \sin t = \frac{\sin t}{25D - 1} \times \frac{25D + 1}{25D + 1}$$

$$= \frac{\sin t (25D + 1)}{625D^2 - 1} = \frac{(25D + 1) \sin t}{-626}$$

$$\boxed{P.I = \frac{25 \cos t + \sin t}{-626}}$$

$$\frac{di}{dt} + 25i = \sin t$$

$$\text{Let, } \frac{d}{dt} = D$$

$$Di + 25i = \sin t$$

$$(D+25)i = \sin t$$

$$\text{A.E, } m+25=0$$

$$m = -25$$

$$\text{C.F} = C_1 e^{-25t}$$

$$P.I = \frac{1}{D+25} \times \sin t \times \frac{D-25}{D-25}$$

$$= \sin t (D-25)$$

$$D^2 - 625$$

$$= \sin (D-25) \sin t$$

$$-626$$

$$P.I = \frac{\cos t - 25 \sin t}{-626}$$

$$i = C.F + P.I$$

$$i = C_1 e^{-25t} + (\cos t - 25 \sin t)$$

$$-626$$

$$\text{at } t=0, i=0$$

$$0 = C_1 - (1-0)$$

$$-626$$

$$C_1 = \frac{1}{626}$$

$$-626$$

$$i = \frac{1}{626} e^{-25t} - (\cos t - 25 \sin t)$$

$$-626$$

Ans

Q. solve, $xy'' - y' + 4x^3y = x^5$.
Given eqⁿ divide by x .

$$y'' - \frac{1}{x} y' + 4x^2 y = x^4.$$

$$P = -\frac{1}{x}, \quad Q = 4x^2, \quad R = x^4$$

$$\left(\frac{dz}{dx}\right)^2 = Q = 4x^2.$$

$$\therefore \frac{dz}{dx} = 2x.$$

$$\int dz = \int 2x dx$$

$$(z = x^2)$$

$$P_1 = P \frac{dz}{dx} + \frac{d^2 z}{dx^2} = -\frac{1}{x} \times 2x + 2 = 0$$

$$P_1 = 0$$

$$Q_1 = Q = \frac{4x^2}{(dz/dx)^2} = \frac{4x^2}{4x^2} = 1$$

$$R_1 = R / (dz/dx)^2 = \frac{x^4}{4x^2} = \frac{x^2}{4}$$

$$\Rightarrow \frac{d^2 y}{dz^2} + 0 + y_1 = \frac{x^2}{4}$$

$$\frac{d^2 y}{dz^2} + y = \frac{x^2}{4} = \frac{z}{4}$$

$$\checkmark \text{ let } \frac{d}{dz} = D$$

$$(D^2 + 1)y = \frac{x^2}{4}$$

$$A.E, \quad m^2 + 1 = 0$$

$$m = \pm i$$

$$C.F = C_1 \cos x + C_2 \sin x$$

$$P.I = \frac{1}{D^2+1} \times \frac{z^2}{4} = \frac{1}{4} [1+D^2]^{-1} \cdot z^2$$

$$= \frac{1}{4} [1-D^2] z^2$$

$$= \frac{1}{4} [z^2 - 0] = \frac{z^2}{4}$$

$$P.I = \frac{z}{4}$$

$$y = C.F + P.I = C_1 \cos z + C_2 \sin z + \frac{z}{4}$$

Replace, z by x^2

$$y = C_1 \cos x^2 + C_2 \sin x^2 + \frac{x^2}{4}$$

Ans

Q Solve, $y'' - 2 \tan x y' + 5y = e^x \sec x$.

$$\frac{d^2 y}{dx^2} - 2 \tan x y' + 5y = e^x \sec x$$

$$P = -2 \tan x \quad Q = 5y \quad R = e^x \sec x$$

$$u = e^{-\int P dx} = e^{\int 2 \tan x dx}$$

$$u = e^{2 \tan x dx} = e^{2 \log \sec x}$$

$$(u = \sec^2 x)$$

$$I = Q - \frac{1}{2} \frac{dP}{dx} - \frac{P^2}{4} = 5 + 1 + 2 \sec^2 x - \frac{4 \tan^2 x}{4}$$

$$\frac{d^2 v}{dx^2} + IV = S \quad (I = 6)$$

$$S = R$$

$$S = e^x \sec x$$

$$\frac{d^2 v}{dx^2} + 6v = e^x$$

$$(S = e^x)$$

$$(D^2 + 6)V = e^x$$

$$A.E = m^2 + 6 = 0$$

$$m^2 = -6$$

$$m = \pm \sqrt{6} i$$

$$C.F = C_1 \cos \sqrt{6} x + C_2 \sin \sqrt{6} x$$

$$P.I = \frac{1}{D^2 + 6} \times e^x = \frac{1}{7} e^x$$

$$\therefore V = C.F + P.I$$

$$V = C_1 \cos \sqrt{6} x + C_2 \sin \sqrt{6} x + \frac{e^x}{7}$$

$$y = uv$$

$$y = \sec x \left[C_1 \cos \sqrt{6} x + C_2 \sin \sqrt{6} x + \frac{e^x}{7} \right]$$

Ans

Q3

by V.O.I, solve,

$$(D^2 - 1)y = 2(1 - e^{-2x})^{-1/2} \quad P=0, Q=-1, R=2(1 - e^{-2x})^{-1/2}$$

$$A.E; m^2 - 1 = 0$$

$$m = \pm 1$$

$$C.F = C_1 e^x + C_2 e^{-x}$$

$$y_1 = e^x \quad y_2 = e^{-x}$$

$$u = - \int \frac{2(1 - e^{-2x})^{-1/2} \times e^{-x}}{-e^x e^{-x} + e^{-x} e^x} dx$$